Discontinuous Numerical Perturbation Reconstructing Algorithm for Convective-Diffusion Equation

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A discontinuous numerical perturbation algorithm, i.e. discontinuous small parameter expansion algorithm is presented in this paper. This algorithm is applied to reconstruct finite difference (FD) and finite volume (FV) schemes for the convective-diffusion (CD) equation. As an example, the CD integral equation and its second-order center FV scheme (call it 2CVS) can be written as,

$$\int_S \rho\phi\vec{u}\cdot\vec{n} = \int_s \mu\nabla\phi\cdot\vec{n}ds, \quad \sum_{j=1}^{J}\left[\frac{\mu}{d_j^2}\vec{d}_j\cdot\vec{S}_j(\phi_{jp}-\phi_p) - m_{jf}\phi_{jf}\right] = 0 \tag{1}$$

repectively. Where S is the surface enclosing control volume (CV), $\vec{n}$ is the unit vector orthogonal to S and directed outwards, $\vec{u}$ is the fluid velovity, ρ is the density, μ is the diffusion coefficient, ϕ represents any transported variable, ϕ_p is the value of ϕ at the CV center (node p), ϕ_{jp} is the value of ϕ at the adjacent CV center (node jp). A typical cell face labeled "jf" (See Fig. 1) is analyzed, $j = 1, 2, \cdots, J$. ϕ_{jf} is the value of ϕ at the cell jf-face center, $\vec{S}_j$ is the aera vector of the cell jf-face and its direction agrees with $\vec{n}$, $\vec{d}_j$ is the vector linking node p and node jp with the direction from p to jp, $d_j = |\vec{d}_j|$, m_{jf} is the mass flux through the cell jf-face, $m_{jf} = \rho U_{\xi_j} S_j$, U_{ξ_j} is the fluid velocity component in the local coordinate direction $\vec{\xi}_j$.

The procedure of discontinuous numerical perturbation is that the scheme (1) is split spatially into J schemes and its j-th scheme ($j = 1, 2, \cdots, J$) is split into upstream scheme for the node p (if $U_{\xi_j} > 0$) and downstream scheme for the node jp and then m_{jf} is reconstructed discontinuously as $G_j^- m_{jf}$ in upstream scheme and $G_j^+ m_{jf}$ in downstream scheme, respectively, and then we have

$$\left[\frac{\mu}{d_j^2}\vec{d}_j\cdot\vec{S}_j - (1-\delta_j)m_{jf}G_j^+\right]\phi_{jp} = \left[\frac{\mu}{d_j^2}\vec{d}_j\cdot\vec{S}_j + \delta_j m_{jf}G_j^-\right]\phi_p \tag{2}$$

where $G_j^+ = 1 + \sum_{n=1}^{N} a_n^+(\delta_j d_j)^n$, $G_j^- = 1 + \sum_{n=1}^{N} a_n^-(1-\delta_j)^n d_j^n$, G_J^+ and G_j^- are called

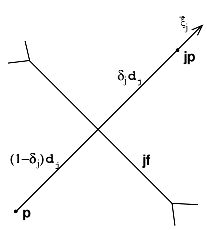

Figure 1: CV center (node p) and adjacent CV center (node jp) and local coordinate.

perturbational reconstruction functions. The coefficients a_n^+ and a_n^- are obtained by eliminating truncation error terms including in the modified differential equation of the scheme (2) and have

$$G_j^+ = 1 + \sum_{n=1}^{N} \frac{(-\delta_j)^n}{1-\delta_j} \left[\frac{1}{n!} - \frac{2n+1}{(n+1)!}\delta_j \right] R_{d_j}^n, G_j^- = 1 + \sum_{n=1}^{N} \frac{(1-\delta_j)^n}{\delta_j} \left[\frac{2n+1}{(n+1)!}\delta_j - \frac{n}{(n+1)!} \right] R_{d_j}^n \quad (3)$$

where $R_{d_j} = \rho U_{\xi_j} d_j/\mu$ can be regarded as the cell (or grid) Reynolds number. The FV scheme (call it discontinuous perturbation FV scheme, DPVS) obtained by discontinuous numerical perturbation reconstruction is

$$\sum_{j=1}^{J} \frac{\mu S_j}{d_j} \left\{ \left[1 - R_{d_j} G_j^+ (1-\delta_j) \right] \phi_{jp} - \left[1 + R_{d_j} G_j^- \delta_j \right] \phi_p \right\} = 0 \quad (4)$$

The present DPVS (4) has the same structure and simplicity as the original 2CVS. Analysis shows that, if N is an odd number and $\frac{N}{2N+1} \le \delta_j \le \frac{N+1}{2N+1}$, the DPVS (4) are absolute positive schemes for any value of R_{d_j} and its interpolation approximate accuracy is $(n+2)$-order. In other cases the DPVS (4) are conditional positive schemes, i.e. conditional stability schemes. Numerical tests solving one- to three-dimensional linear CD equations, one and two dimensional Burgers equations show that the averaged errors of the present DPVS (4) are much smaller than those of the second-order center FV scheme (2CVS) in all tested cases, and the present DPVS (4) with odd number order accurate do not yield oscillatory solution, while 2CVS and the DPVS (4) with even number order accurate oscillate in coarse grid cases. The averaged errors of the present DPVS (4) are also smaller than those of WENO and Discontinuous Galerkin (DG) schemes if all three schemes have the same accuracy order.

Similarly, some higher-order accuracy, absolute stability center finite difference schemes (call them discontinuous perturbation finite difference schemes, DPDS) are also obtained by discontinuous perturbation reconstructing the second-order accurate center FD scheme (2CDS) for the convective diffusion equation. DPDS's excellent properties are verified by analysis and numerical tests. As an example, Table 1 gives the maximum errors L_∞ and averaged errors L_1 of the second-order central difference scheme (2CDS), the third order DPDS (3DPDS) and the fifth order WENO scheme (5WENO) solving the linear transport equation and Burgers equation, respectively. It should be mentioned that 3DPDS is a central scheme using only three nodes, has the same structure and simplicity as 2CDS, and does not need artificial viscosity or limiter, while 5WENO is an upstream scheme using seven nodes and requires weight functions.

Table 1: Comparison of different schemes

	N	2CDS		3DPDS		5WENO	
		L_∞	L_1	L_∞	L_1	L_∞	L_1
Linear	160	—-	—-	0.6940e+0	0.4872e-2	0.2631e+1	0.2034e-1
	320	0.4141e+0	0.1861e-2	0.1487e-1	0.7443e-4	0.3559e+0	0.1960e-2
	640	0.4288e-1	0.2196e-3	0.4190e-3	0.2224e-5	0.3964e-1	0.2177e-3
	1280	0.6930e-2	0.3746e-4	0.1751e-4	0.9516e-7	0.6983e-2	0.3815e-4
Burgers	160	—-	—-	0.1213e+0	0.1779e-2	0.1205e+0	0.1646e-2
	320	—-	—-	0.4251e-2	0.4158e-4	0.1166e+0	0.9521e-3
	640	0.1278e+0	0.6278e-3	0.4036e-1	0.1780e-3	0.2749e-1	0.2172e-3
	1280	0.2440e-1	0.1442e-3	0.9744e-2	0.5011e-4	0.3633e-2	0.2952e-4