Numerical modelling of flow in lower urinary tract using high-resolution methods

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Abstract: In this paper we propose a new numerical scheme based on the finite volumes to simulate the urethra flow and compare its properties with the traditional schemes for solving hyperbolic balance laws. Our approach is based on the Riemann solver designed for the augmented quasilinear homogeneous formulation. The scheme has general semidiscrete wave–propagation form and can be extended to arbitrary high order accuracy. The main goal is to construct the scheme, which is well balanced, i.e. maintains not only some special steady states but all steady states which can occur.

Keywords: Balance Laws, Steady States, Positive Semidefiniteness, Urethra

1 Introduction

We briefly introduce a problem describing fluid flow through the elastic tube represented by hyperbolic partial differential equations with the source term. We are interested in the mathematical model describing the fluid flow through the male urethra. It is based on balance law $\mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x = \boldsymbol{\psi}(\mathbf{u}, x)$ describe in [1] which has the following form

$$\begin{aligned} a_t + q_x &= 0, \\ q_t + \left(\frac{q^2}{a} + \frac{a^2}{2\rho\beta}\right)_x &= \frac{a}{\rho}\left(\frac{a_0}{\beta}\right)_x + \frac{a^2}{2\rho\beta^2}\beta_x - \frac{q^2}{4a^2}\sqrt{\frac{\pi}{a}}\lambda(Re), \end{aligned} \tag{1}$$

where $a = a(x,t)$ is the unknown cross-section area, $q = q(x,t)$ is the unknown flow rate (we also denote $v = v(x,t)$ as the fluid velocity, $v = \frac{q}{a}$), ρ is the fluid density, $a_0 = a_0(x)$ is the cross-section of the tube under no pressure, $\beta = \beta(x,t)$ is the coefficient describing tube compliance and $\lambda(Re)$ is the Mooney-Darcy friction factor ($\lambda(Re) = 64/Re$ for laminar flow). Re is the Reynolds number defined by

$$Re = \frac{\rho q}{\mu a}\sqrt{\frac{4a}{\pi}}, \tag{2}$$

where μ is fluid viscosity. This model contains constitutive relation between the pressure and the cross section of the tube

$$p = \frac{a - a_0}{\beta} + p_e, \tag{3}$$

where p_e is surrounding pressure.

2 Well-Balance Property

Well-balanced schemes are requested in the problems where the steady states can occur. The general steady state means $\mathbf{u}_t = \mathbf{0}$. Therefore, it is necessary to balance flux difference and the source terms

$$[\mathbf{f}(\mathbf{u})]_x = \boldsymbol{\psi}(\mathbf{u}, x). \tag{4}$$

There will be shown several ways how to construct numerical methods for preserving steady states. We will introduce the main ideas of some of them. Well-balanced schemes satisfy discrete analogy of property (4) by suitable discretization of the flux difference and the approximation of the source terms. At the same time, we will discussed the positive semidefiniteness of used scheme, which is also an important property for urethra flow.

2.1 Decompositions based on augmented system

This procedure is based on the extension of the system (1) by other equations. This was derived in [2] for the shallow water flow. The advantage of this step is in the conversion of the nonhomogeneous system to the homogeneous one. In our problem (1) we get nonconservative system $\mathbf{w}_t + \mathbf{B}(\mathbf{w})\mathbf{w}_x = \mathbf{0}$, where $\mathbf{w} = [a, q, av^2 + \frac{a^2}{2\rho\beta}, \frac{a_0}{\beta}, \beta]^T$.

This system has five linearly independent eigenvectors. The approximation is chosen to be able to prove the consistency and provide the stability of the algorithm. In some special cases this scheme is conservative and we can guarantee the positive semidefiniteness, but only under the additional assumptions.

This augmented system is solved by the high order numerical scheme in the fluctuation form

$$\frac{\partial \mathbf{Q}_j}{\partial t} = -\frac{1}{\Delta x}[\mathbf{A}^-(\mathbf{Q}^-_{j+1/2}, \mathbf{Q}^+_{j+1/2}) + \mathbf{A}(\mathbf{Q}^-_{j+1/2}, \mathbf{Q}^+_{j-1/2}) + \mathbf{A}^+(\mathbf{Q}^-_{j-1/2}, \mathbf{Q}^+_{j-1/2})], \tag{5}$$

where $\mathbf{A}^{\pm}(\mathbf{Q}^-_{j+1/2}, \mathbf{Q}^+_{j+1/2})$ are so called fluctuations, which are derived by the decomposition of eigenvectors of augmented system.

The steady state for the augmented system means $\mathbf{B}(\mathbf{w})\mathbf{w}_x = \mathbf{0}$, therefore $\mathbf{w}_x$ is a linear combination of the eigenvectors corresponding to the zero eigenvalues. We derive the approximations of the eigenvectors of the matrix $\mathbf{B}(\mathbf{w})$ to preserves general steady state.

3 Conclusion and Future Work

We will present new numerical scheme in wave–propagation form for solving fluid flow problems with the source terms (i.e. the problems in quasilinear form). This scheme is well balanced - maintains not only some special steady states but all steady states which can occur. This scheme can be extended to the arbitrary order. We will present the comparison of this scheme with the traditional other ones and declare its properties on the numerical experiments. We also prepare implementation of boundary conditions using the exact solution of the Riemann problem.

References

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