Three-dimensional thin film flow over topography: full Navier-Stokes solutions

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Abstract: The problem of steady-state gravity-driven film flow over topography is explored via solutions of the the full Navier-Stokes system of equations. Besides providing a unique benchmark with which to assess the limitations of related models based on the long-wave approximation, a defining feature is the lifting of many of the constraints associated with the latter while in addition enabling the internal flow topology to be explored as the Reynolds number is varied.

Keywords: Thin films, free surface flow, finite elements, flow topology

1 Introduction

Investigating the problem of thin film flow over topography, based on the solution of the full Navier-Stokes (N-S) equations, has over the years been restricted to two-dimensional problems, due in the main to the computational resource and effort required [1]. To do so, however, overcomes many of the restrictions associated with simpler models based on the long-wave approximation, removing in one go any constraints, other than physically realisable ones, concerning choice of capillary number, film thickness or topography aspect ratio; in addition the internal velocity field forms part of the solution and hence the internal flow topology is revealed.

2 Problem Specification and Method of Solution

The problem of interest consists of a continuous thin liquid film, of asymptotic thickness H_0, flowing down a substrate, containing a trench topography and inclined at an angle θ to the horizontal, the volumetric flow rate being Q_0 per unit width and the characteristic velocity $U_0 = 3Q_0/2H_0$. The fluid is considered to be incompressible with constant density, ρ, viscosity, μ, and surface tension, σ – see [1] or [2].

The strategy adopted to solve the N-S equations governing the three-dimensional flow involves mixed-interpolation using a Bubnov-Galerkin finite element scheme [3], free-surface parametrisation based on the Arbitrary Lagrangian-Eulerian method of spines [4], the use of a direct parallel multifrontal method [5], as contained in the MUMPS library [6], and utilisation of the memory-efficient out-of-core approach for storing matrix cofactors on the hard drive. The three-dimensional solutions obtained in this way represent the first of their kind, enabling both the internal flow topology and free-surface disturbance generated to be explored simultaneously.

3 Results

Analysing the internal flow topology which arises, reveals that across the trench topography the three-dimensional thin film flow is topologically different internally from its two-dimensional counterpart as reported in [1]: whereas for the latter eddy centres are always elliptic points, for the former, at the mid-plane, they may instead be foci. In addition, the three-dimensional flow case leads to different flow topologies, dependent not only on the trench geometry but also on the capillary, $\mathrm{Ca} = \mu U_0/\sigma$, and Reynolds, $\mathrm{Re} = \mu U_0 H_0/\rho$, number; Figure 1 is a typical case. For $\mathrm{Re} = 0$ the trajectories of the flow inside the trench form closed eddies; for $\mathrm{Re} = 10$, the trajectories encroach into the trench near the left hand symmetry mid-plane (see the blue and red trajectories) swirl around several times during which they are laterally displaced away from the mid-plane, they exit the trench close to its (right hand) side before travelling downstream.

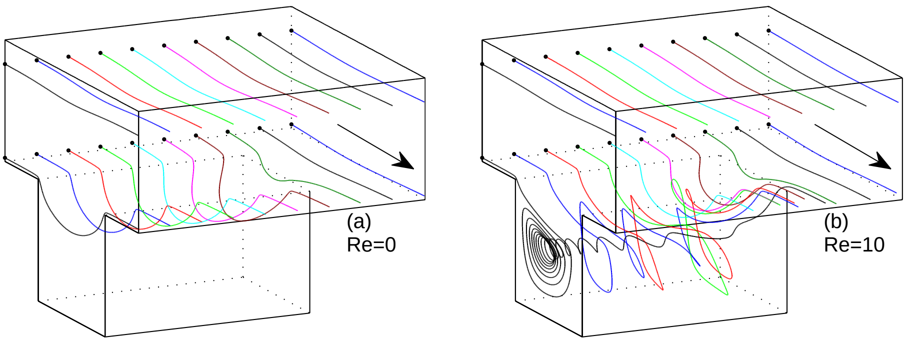

Figure 1: Three-dimensional flow topology, obtained by integrating along path lines, for the case of a localised rectangular trench topography with nondimensional (scaled with respect to H_0) streamwise length $l_t = 1.5$, spanwise width $w_t = 3.0$, depth $s_0 = 1.0$ and capillary number $\mathrm{Ca} = 10^{-3}$; $\theta = 30^o$: $\mathrm{Re} = 0$ (left) and $\mathrm{Re} = 10$ (right). Starting positions are denoted as filled circles located adjacent to the substrate at $z = 0.03$ and to the free surface at $z = 0.8$. For illustrative purposes different colours are used for streamlines corresponding to different starting positions. The symmetry mid-plane through the centre of the trench is on the left hand side, the closed side of the trench is on the right. The arrow shows the direction of flow.

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