# Analytical solution for singularities in Stokes flow and applications to finite element solution of Navier-Stokes equations with high precision

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**Abstract:** We present analytical solution of the Stokes problem in 2D domains. This is then used to find the asymptotic behavior of the solution in the vicinity of corners, also for Navier-Stokes equations in 2D. We apply this to construct numerical finite element solution with high precision.

*Keywords:* Stokes problem, Finite elements, Asymptotic behaviour.

**1. Introduction** The behaviour of the solution of Stokes and Navier-Stokes equations in domains with corners or with discontinuities in boundary conditions is still not quite well understood. We use the analytical solution to characterize the singular part of the solution. The asymptotics apply also to Navier-Stokes equations. The results are applied to two examples: the flow in a channel with forward and backward steps, and the problem of lid driven cavity.

## 2. Analytical solution of the Stokes flow near corners

We consider the Stokes problem in vorticity - stream function formulation, cf [1], and transform the problem to polar coordinates $x = r\cos\vartheta$ , $y = r\sin\vartheta$ , with the pole in the corner $P$, cf. Fig. 1.

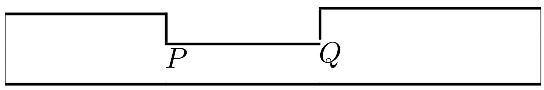



Figure 1: The solution domain $\Omega$.

So we have to find stream function $\psi(r,\vartheta)$ and vorticity $\omega(r,\vartheta)$, satisfying the equations

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\vartheta^2} = -\omega , \qquad \frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega}{\partial\vartheta^2} = 0 . \tag{1}$$

We solve the equations (1) by means of separation of variables,

$$\psi(r,\vartheta) = P(r)\cdot F(\vartheta) , \quad \omega(r,\vartheta) = R(r)\cdot G(\vartheta) . \tag{2}$$

Analyzing arizing differential equations we come to the asymptotic formula for stream function

$$\psi(r,\vartheta) = r^{-\sqrt{\varkappa}+2}\cdot F(\vartheta) \quad (+h.o.t.), \tag{3}$$

where $\varkappa$ is a positive parameter depending only on the angle of the corner.

**Example 1**. We consider flow in 2D region with boundary corner of internal angle $\varphi$, as e.g. on Fig. 1. We assume nonslip boundary conditions, so the boundary conditions for the stream function are

$$\psi(r,0) = 0, \quad \psi(r,\varphi) = 0, \quad \frac{\partial\psi}{\partial\vartheta}(r,0) = 0, \quad \frac{\partial\psi}{\partial\vartheta}(r,\varphi) = 0. \tag{4}$$

As an example we take the domain shown in Fig. 1, where the angle $\varphi = \frac{3}{2}\pi$. Then we get $\sqrt{\varkappa} = 0.45552$. Now, by (3) we get e.g. the asymptotics for stream function, near the corner P

$$\psi(r,\vartheta) = r^{1.54448} \cdot F(\vartheta) \quad (+h.o.t.), \tag{5}$$

with $F$ independent of $r$. So we get the asymptotics for velocity components and pressure

$$u_r = r^{0.54448}\, F_1(\vartheta), \quad u_\vartheta = r^{0.54448}\, F_2(\vartheta), \quad p = r^{-0.45552}\, F_3(\vartheta), \tag{6}$$

where $F_1(\vartheta), F_2(\vartheta), F_3(\vartheta)$ are independent of $r$. The same formulas apply to point $Q$.

**Example 2.** Let us now consider 2D flow in lid driven cavity with boundary conditions

$$\psi(r,0) = 0, \quad \psi(r,\varphi) = 0, \qquad \frac{1}{r}\frac{\partial\psi}{\partial\vartheta}(r,0) = 1, \quad \frac{1}{r}\frac{\partial\psi}{\partial\vartheta}(r,\varphi) = 0, \tag{7}$$

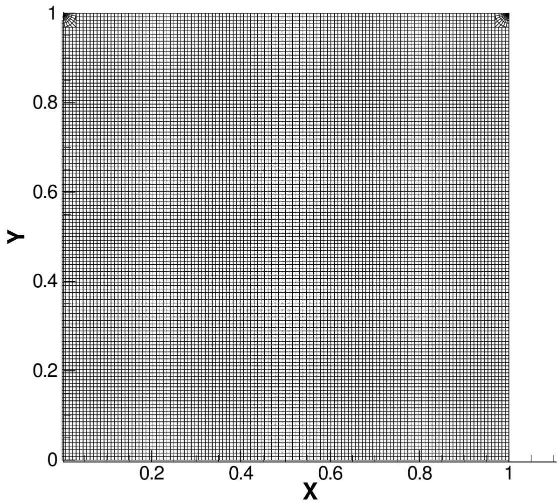
where $\varphi = -\frac{3}{2}\pi$. We solve the equations (1) similarly as above, by means of separation (2). One can then derive the asymptotics in upper corners of the cavity

$$\psi(r,\vartheta) = r \cdot F(\vartheta), \quad u_r = F'(\vartheta), \quad u_\vartheta = F(\vartheta), \quad p(r,\vartheta) = \frac{1}{r}\Phi(\vartheta), \tag{8}$$

which are much worse than in case of the corner in Example 1.

## 3. Application to finite element calculations

The application of the asymptotics may be at least twofold. First, the analytical solution near corners may be used to direct checking of numerical solution. Second, combining the asymptotics of Navier-Stokes equations with a priori estimates we get an algorithm for generating the finite element mesh at such corners cf. [2], [3]. As an application we show on Fig. 2 the locally refined mesh near upper corners of lid driven cavity, and pressure calculated by this algorithm.



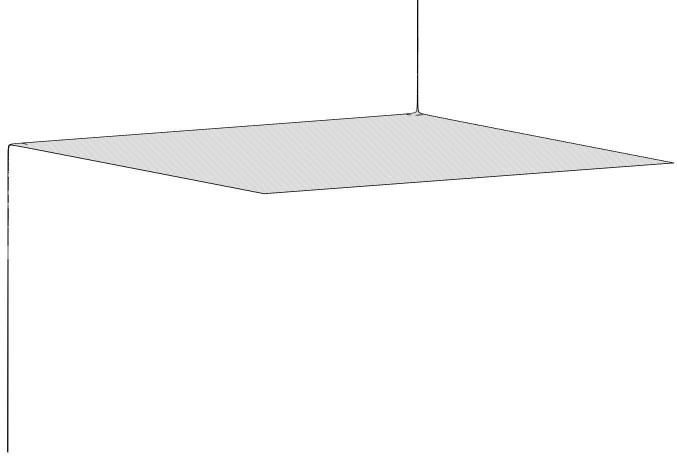



Figure 2: Lid driven cavity, Re = 10,000 Left: mesh 128×128 refined locally Right: pressure

## 4. Conclusion

We solve analytically the Stokes problem in 2D domains, using polar coordinates and separation of variables. This is then used to find the asymptotics of the solution near corners, also for Navier-Stokes equations. We show application to very precise finite element solution.

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# References

[1] G. K. Batchelor. *An Introduction to Fluid Dynamics.* Cambridge University Press. 1967.

[2] P. Burda, J. Novotný and J. Šístek. Precise FEM solution of a corner singularity using an adjusted mesh. *Int. J. Numer. Meth. Fluids*, **47**, 1285–1292 2005.

[3] P. Burda, J. Novotný and J. Šístek, Singularities in lid driven cavity solved by adjusted finite element method, *In: Computational Fluid Dynamics 2011, A. Kuzmin (Ed.)*, Springer-Verlag, 799 - 805, 2011.