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A Monolithic Approach for the Solution of the Incompressible Magnetohydrodynamics Equations in Two- and Three-Dimensions

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Abstract: A numerical algorithm has been developed to solve the incompressible magnetohydrodynamics (MHD) equations in a fully coupled form in two- and three-dimensions. The numerical approach is based on the side centered unstructured finite volume approximation where the velocity and magnetic filed vector components are defined at the edge/face midpoints, meanwhile the pressure term is defined at the element centroids. In order to enforce a divergence free magnetic field, a magnetic pressure is introduced to the induction equation. The resulting large-scale algebraic linear equations are solved using a one-level restricted additive Schwarz preconditioner with a block-incomplete factorization within each partitioned sub-domains. The parallel implementation of the present fully coupled unstructured MHD solver is based on the PETSc library for improving the efficiency of the parallel algorithm. The numerical algorithm is validated for 2D and 3D lid-driven cavity flows with insulating walls.

Keywords: Incompressible magnetohydrodynamics, semi-staggered finite volume method, monolithic approach, lid-driven cavity.

1 Introduction

Magnetohydrodynamics (MHD) deals with the interaction between magnetic field and the fluid flow. The fluid has to be electrically conducting and non-magnetic in order to interact with the magnetic field. The interaction is a result of Ampere's and Faraday's laws as well as the Lorentz force. The interaction leads to an electromotor force (emf) due to the relative motion of the fluid within the magnetic field and electrical currents are induced. Then, these currents induce a secondary magnetic field and the combined magnetic fields interacts with the induced current density, leading to the Lorentz force. Magnetohydrodynamics is important for many applications in engineering and scientific phenomenon such as sunspots, solar flares, interaction between solar winds and Earth's magnetosphere, controlled thermofusion, propulsion, electromagnetic pumps, control of liquid metals, etc. [1, 2].

The mathematical description of incompressible MHD flow includes the conservation of mass, the conservation of momentum, the magnetic induction equation and the divergence-free condition of the magnetic field. The resulting system of the MHD equations can be solved using two different coupling strategies. The first one is based on partitioned (staggered) methods in which the equations for fluid and magnetic fields are solved separately. The other category is the fully coupled (monolithic) methods. In monolithic approaches the equations for both fields are discretized and solved simultaneously. Staggered approaches provide the freedom to choose optimized solvers for each unknown field. But their convergence rate are slow for fixed point (Picard) iterations and they may diverge for strong interactions (i.e Hartman number greater than unity). The advantage of the monolithic approaches is their robustness, but this also leads to computational expense because they require the solution of large systems of coupled non-linear equations. The comparison of both methods can be found in [3].

In the present work, the impressible MHD equations are solved in a fully coupled (monolithic) manner using the divergence-free side centered finite volume approximation, where the velocity and magnetic filed vector components are defined at the edge/face midpoints, meanwhile the pressure term is defined at the element centroids. In order to solve the overdetermined system, a magnetic pressure is introduce at the cell center and the gradient of this pressure is added to the magnetic induction equation with certain boundary conditions as described in [4] in order to enforce divergence-free magnetic field. The resulting algebraic system is solved in a fully coupled manner using a one-level restricted additive Schwarz preconditioner provided by PETSc library [5] with a block-incomplete factorization within each partitioned sub-domains.

2 Mathematical and Numerical Formulation

The non-dimensional conservation form of the governing equations for the incompressible magnetohydrodynamics (MHD) flow are given as follows:

$$Re\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \left[Re\mathbf{u} \otimes \mathbf{u} - SRe\mathbf{B} \otimes \mathbf{B} + \left(p + SRe\frac{B^2}{2} \right) \mathbf{I} - \mathbf{T} \right] = 0 \tag{1}$$

$$Re_m \frac{\partial \mathbf{B}}{\partial t} - \nabla^2 \mathbf{B} + Re_m \nabla \cdot [-\mathbf{u} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{u}] = 0 \tag{2}$$

The non-dimensional Reynolds number (Re), magnetic Reynolds number (Re_m) and coupling parameter (Stuart number) (S) are defined as follows

$$Re = \frac{\rho U L}{\mu_f}, \quad Re_m = \mu_m \sigma U L, \quad S = \frac{B^2}{\rho \mu_m U^2}$$

where $\mathbf{u}$ is the velocity vector, $\mathbf{B}$ is the magnetic field, $\mathbf{T}$ is the fluid stress tensor, $\mathbf{I}$ is identity matrix, ρ is the fluid density, p is the pressure, σ is the electrical conductivity, μ_m is the magnetic permeability and μ_f is the dynamic viscosity of the fluid. In order to satisfy the solenoidal property of magnetic field, the gradient of a Lagrange multiplier q introduced to the magnetic induction equation as proposed in [4]. Therefore, the integral form of incompressible MHD equations over a control volume Ω with boundary $\partial\Omega$ can be written in Cartesian coordinate system in dimensionless form as follows:

Continuity equation:

$$-\oint_{\partial\Omega_e} \mathbf{n} \cdot \mathbf{u} dS = 0 \tag{3}$$

Momentum equation:

$$\begin{aligned} Re\int_{\Omega_d} \frac{\partial \mathbf{u}}{\partial t} dV + Re\oint_{\partial\Omega_d} [\mathbf{n} \cdot \mathbf{u}]\mathbf{u} dS + \oint_{\partial\Omega_d} \mathbf{n} P dS \\ - \oint_{\partial\Omega_d} \mathbf{n} \cdot \nabla \mathbf{u} dS - SRe\oint_{\partial\Omega_d} [\mathbf{n} \cdot \mathbf{B}]\mathbf{B} dS = 0 \end{aligned} \tag{4}$$

Magnetic induction equation:

$$\begin{aligned} Re_m \int_{\Omega_d} \frac{\partial \mathbf{B}}{\partial t} dV + Re_m \oint_{\partial\Omega_d} [\mathbf{n} \cdot \mathbf{u}]\mathbf{B} dS + \oint_{\partial\Omega_d} \mathbf{n} q dS \\ -Re_m \oint_{\partial\Omega_d} [\mathbf{n} \cdot \mathbf{B}]\mathbf{u} dS - \oint_{\partial\Omega_d} \mathbf{n} \cdot \nabla \mathbf{B} dS = 0 \end{aligned} \tag{5}$$

Gauss' law of magnetism states that $\mathbf{B}$ is solenoidal:

$$-\oint_{\partial\Omega_e} \mathbf{n} \cdot \mathbf{B} dS = 0 \tag{6}$$

In here, P is defined as follows

$$P = \left[p + S\frac{B^2}{2}\right]$$

In the present study, semi-staggered finite volume formulation in [6] is applied to the solution of incompressible magnetohydrodynamics equations. The discretization of momentum and magnetic induction equation is done over the dual control volume. The continuity equation and the divergence of magnetic field are integrated within the quadrilateral/hexahedral elements. The discretization leads to the following coupled system of algebraic equations:

$$\left[\begin{array}{ccc|ccc|c|c} A_{11} & 0 & 0 & A_{14} & 0 & 0 & A_{17} & 0 \\ 0 & A_{22} & 0 & 0 & A_{25} & 0 & A_{27} & 0 \\ 0 & 0 & A_{33} & 0 & 0 & A_{36} & A_{37} & 0 \\ \hline A_{41} & 0 & 0 & A_{44} & 0 & 0 & 0 & A_{48} \\ 0 & A_{52} & 0 & 0 & A_{55} & 0 & 0 & A_{58} \\ 0 & 0 & A_{63} & 0 & 0 & A_{66} & 0 & A_{68} \\ \hline A_{71} & A_{72} & A_{73} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & A_{84} & A_{85} & A_{86} & 0 & 0 \end{array}\right] \left[\begin{array}{c} u \\ v \\ w \\ \hline B_x \\ B_y \\ B_z \\ \hline P \\ \hline q \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \\ \hline b_4 \\ b_5 \\ b_6 \\ \hline 0 \\ \hline 0 \end{array}\right] \tag{7}$$

where, $A_{11}, A_{22}, A_{33}, A_{44}, A_{55}, A_{66}$ are the convection diffusion operators, $(A_{17}, A_{27}, A_{37}, A_{48}, A_{58}, A_{68})^T$ are the gradient operators and $A_{71}, A_{72}, A_{73}A_{85}, A_{86}, A_{87}$, are the divergence operators. However, it is rather difficult to solve the fully-coupled system due to the zero blocks arising from the divergence free constraints. In order to remove the zero blocks in the original system, an upper triangular right preconditioner is used as follows:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 \\ A_{21} & A_{22} & 0 & A_{24} \\ A_{31} & 0 & 0 & 0 \\ 0 & A_{42} & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & A_{13} & 0 \\ 0 & I & 0 & A_{24} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \\ P \\ q \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

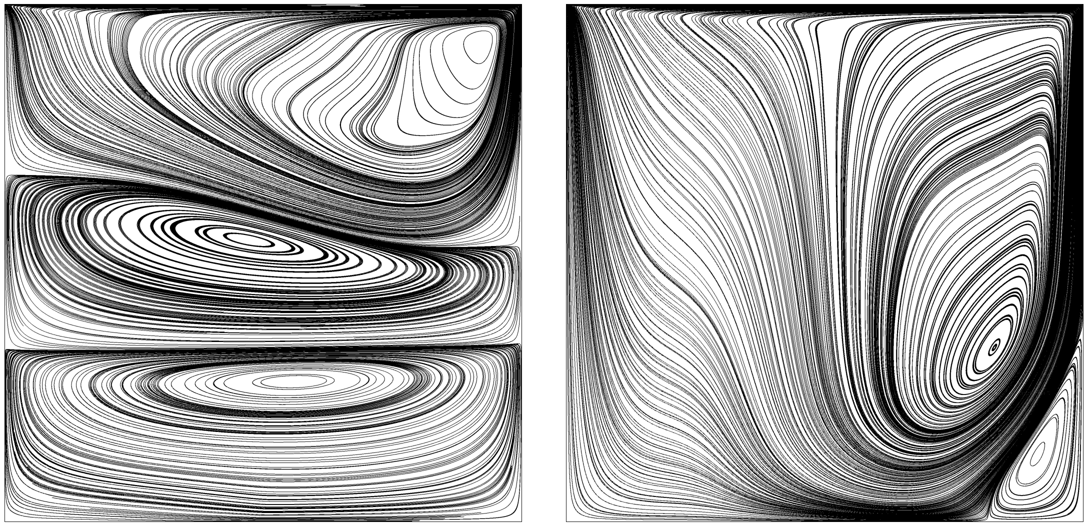
The velocity and magnetic field can be calculates as follows

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{B} \\ P \\ q \end{bmatrix} = \begin{bmatrix} I & 0 & A_{13} & 0 \\ 0 & I & 0 & A_{24} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \\ P \\ q \end{bmatrix} \tag{9}$$

In the modified system, the zero blocks are replaced by $A_{71}A_{17} + A_{72}A_{27} + A_{73}A_{37}$ and $A_{84}A_{48} + A_{85}A_{58} + A_{86}A_{68}$ which are the scaled discrete Laplacians. In this work, one-level restricted additive Schwarz preconditioner with a block-incomplete factorization is used within each partitioned sub-domains for the modified system and the implementation is done by PETSc software package developed at Argonne National Laboratories [5]. METIS library [7] is employed for a balanced domain decomposition.

3 Numerical Results

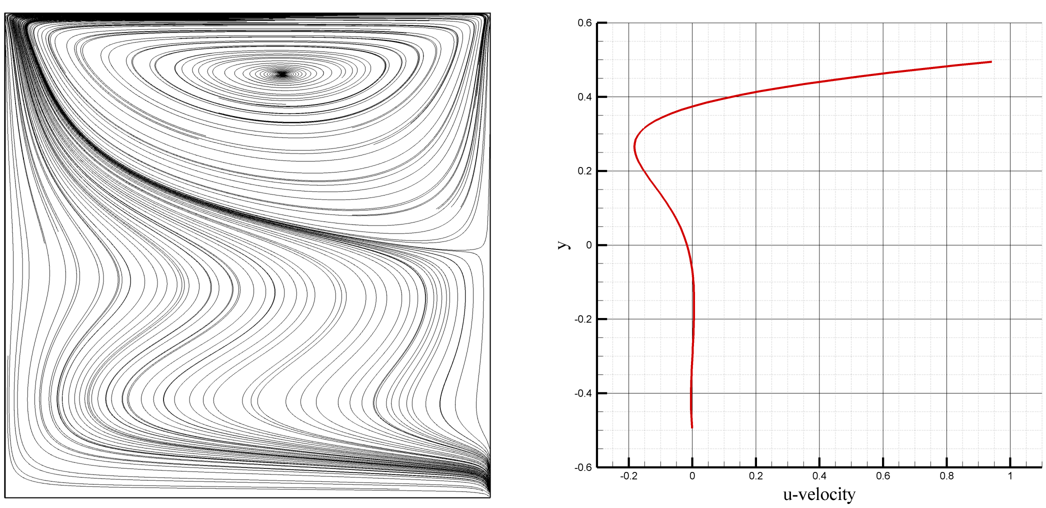
The two-dimensional square cavity problem is solved in order to assess the accuracy of the proposed MHD algorithm. The upper wall is moving in the positive $x-$direction with unit velocity $\mathbf{u} = (1, 0)$. The walls are assumed to be insulating. Two different cases are considered: a horizontal ($\mathbf{B} = (1, 0)$) and a vertical ($\mathbf{B} = (0, 1)$) external magnetic field. For both cases the Reynolds number is set to $Re = 10000$, the Stuart numbers are $S = 1$ for horizontal magnetic field and $S = 0.6426$ for vertical magnetic field. The present results are compared with the work of Marioni et al. [8]. As seen in the Figure 1a, when Stuart number is increased, the flow tends to align with the applied magnetic field and thinner vortices are created. The effect of vertical magnetic field is shown in the Figure 1b. The streamlines tend to align with the applied magnetic field and the primary vortex moves towards to the right wall.



(a) (b)

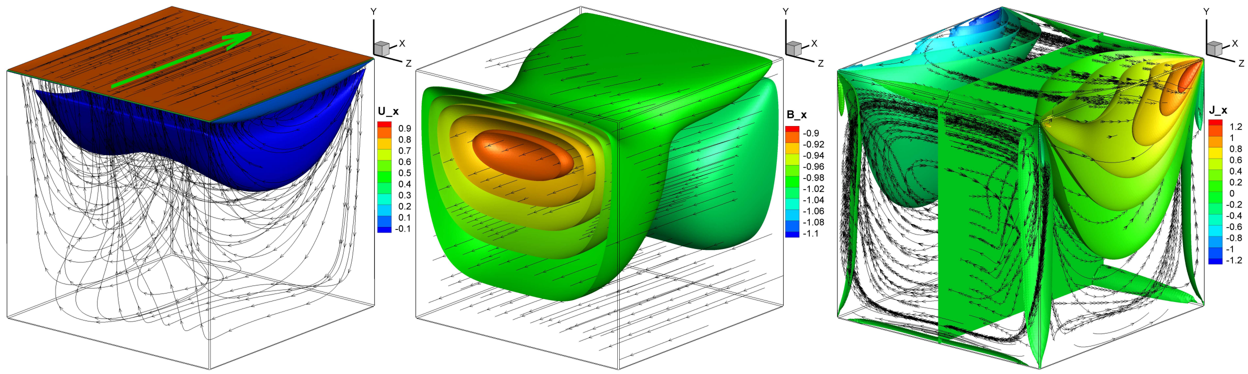
Figure 1: Streamlines for $Re = 10000$ for (a) $\mathbf{B} = (1, 0)$, $S = 1$ and (b) $\mathbf{B} = (0, 1)$, $S = 0.6426$.

The numerical algorithm is also applied to the three-dimensional cubic cavity problem with externally applied magnetic field. Again, the non-dimensional length of cavity is unity and the walls are assumed to be insulated. The upper lid is moving with the velocity $\mathbf{u} = (1, 0, 0,)$ and the external magnetic field vector is $\mathbf{B} = (-1, 0, 0)$. In Figure 2a, the streamlines for $Re = 100$, $Re_m = 10$ and $S = 1$ at $z = 0$ plane are shown. The present results are in good agreement with the work of Phillips et al. [9]. As it may be seen in the Figure 2b, $u-$velocity vector component is almost zero in the lower part of the cavity. The iso-contours of $x-$components of velocity field are shown in Figure 3a and the green arrow indicates the direction of lid movement. Since the magnetic Reynolds number is relatively high, the magnetic field lines close to the lid are bent due to fluid flow as seen in the Figure 3b. Finally, the current density is shown in Figure 3c.



(a) (b)

Figure 2: Streamlines for $Re = 100$, $Re_m = 10$ and $S = 1$ with horizontal magnetic field.



(a) (b) (c)

Figure 3: The x-component of (a) velocity, (b) magnetic field, (c) current density for $Re = 100$, $Re_m = 10$ and $S = 1$ with horizontal magnetic field.

4 Conclusion

In this study, a semi-staggered unstructured finite volume method is developed for the solution of incompressible magnetohydrodynamics equations. In the present approach, the velocity and magnetic filed vector components are defined at the edge/face midpoints, meanwhile the pressure term is defined at the element centroids. The resulting system of algebraic equations is solved in a fully-coupled manner. The divergence-free condition of the magnetic field is satisfied by introducing the gradient of a scalar multiplier into the magnetic induction equation. One-level restricted additive Schwarz method is used for preconditioning and the implementation of the method is done by using PETSc library. The numerical algorithm has been validated for the two and three-dimensional lid-driven cavity problems.

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