Oral presentation | Higher order methods

Higher order methods-II

Tue. Jul 16, 2024 10:45 AM - 12:45 PM Room C

[4-C-01] An unfitted high-order spectral element method for incompressible Navier-Stokes equations with a free surface: The pressure problem

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Keywords: spectral element method, shifted boundary method, incompressible free surface Navier-Stokes, high-order, unfitted mesh

An unfitted high-order spectral element method for the incompressible Navier-Stokes equations with a free surface: The pressure problem

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Financial support for ICCFD12: Thomas B. Thriges Fond and Danish Society for Naval Architecture and Marine Engineering.

Outline

My PhD project.

Motivation of work.

The shifted boundary method:
- In a general numerical setting.
- In the setting of modeling wave propagation and wave-structure interactions.

Numerical results.

Conclusion and perspectives.

Contact information and references.

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About me and my Ph.D. project.

2nd year Ph.D. student.

Ph.D. project: "New advanced simulation techniques for wave energy converters".

Focus on hydrodynamic numerical modelling of wave propagation and wave-structure interactions.

Including:

- Linear and fully nonlinear potential flow formulations by high-order finite element approximations.
- Also, the incompressible Navier-Stokes equations with a free surface. (this talk).

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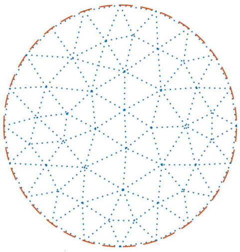
Setting the scene

We want to solve
some fluid-governing equations with a free surface and – potentially – moving bodies and complex geometrical features.

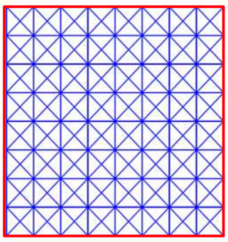
By using
an element based (multi-domain) numerical method, e.g., the finite element method (FEM) or the spectral element method (SEM).

Some terminology:

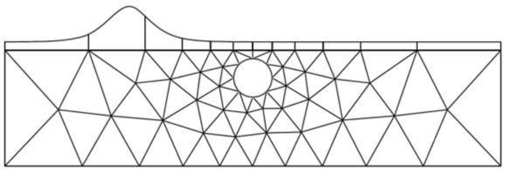


Unstructured mesh of triangular elements (dotted) for curved domain (dashed). Body/ boundary fitted/ conforming.

Structured mesh of triangular elements (blue) for square domain (red). Body/ boundary fitted/ conforming.

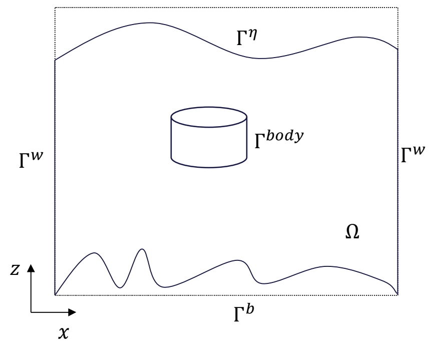


Motivation – Part 1/3


Courtesy of Engsig-Karup et al. (2019).

Domain for the fluid-governing equations:

Symbol:	**Description:**	**Features:**
Ω	Fluid domain	D M C
Γ^{η}	Free surface	D M C
Γ^{b}	Bathymetry	C
Γ^{w}	Wall	
Γ^{body}	Body boundary	M C (D)



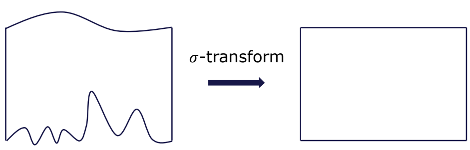
Features:

D: Deforming
M: Moving
C: Complex/curved

Motivation – Part 2/3

Problems:

Meshing in general (complex domains = sophisticated mesh generator, experienced engineer. Takes time).
Re-meshing / mesh updating (computational bottleneck).
Curvilinear meshes (non-trivial to construct for high-order elements).
Relying on transformations compromises simulation possibilities (e.g., σ-transform = no bodies).



A solution:

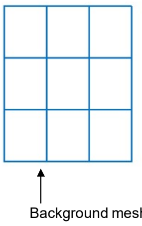
Unfitted/embedded/immersed-type boundary methods (next slide).

Motivation – Part 3/3

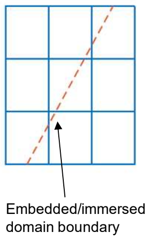
Unfitted/embedded/immersed boundary methods:

Overall idea: Decouple the mesh generation and the domain.

1st step: Generate a simple background mesh.



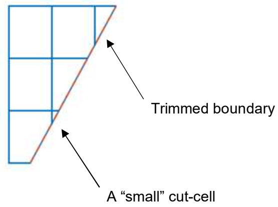
2nd step: Embed/ immerse the domain.



3rd step: Satisfy the boundary conditions:

Forcing terms: Only 1st or 2nd order accurate (problematic).

Cut-cell approaches: Small-time step, ill-conditioning leading to instabilities and lack of convergence. Non-trivial for high-order methods. (again, problematic).



A promising solution: The shifted boundary method (next slide).

The shifted boundary method (SBM) – Part 1/3

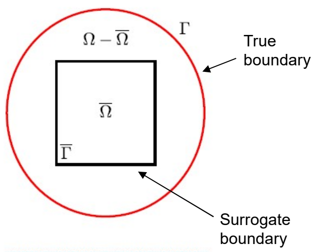
Origin: proposed by Main & Scovazzi (2018a, 2018b) for solving boundary value problems with the FEM.

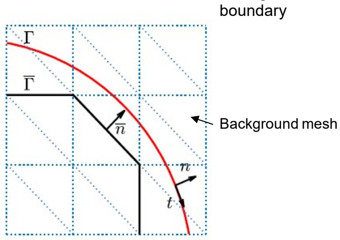
The general idea:

Generate a background mesh.
Embed the ***true*** domain, Ω, into the mesh.
Discard unused elements.
Remaining elements = ***surrogate*** domain, $\overline{\Omega}$.

Similarly, we denote the ***true*** boundary by Γ and the ***surrogate*** boundary by $\overline{\Gamma}$.

Note: Only whole elements (no small-cut-cell-problem).





The shifted boundary method (SBM) – Part 2/3

Time for an animation!

A square background mesh of quadrilaterals.

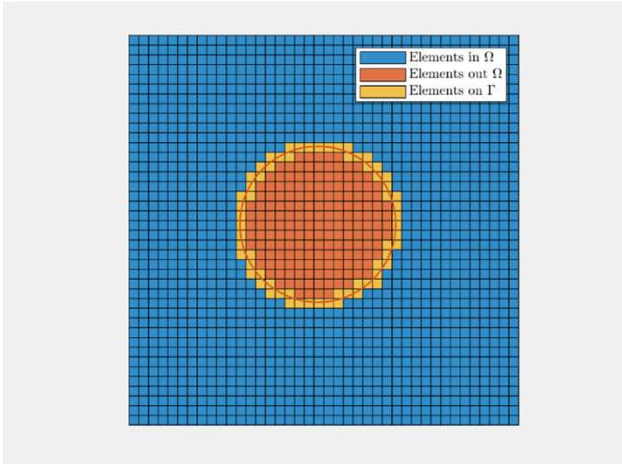
Move a circle around with

» Blue: Elements in Ω.

» Red: Elements out of Ω.

» Yellow: Elements on Γ.

Identification is based on level sets.



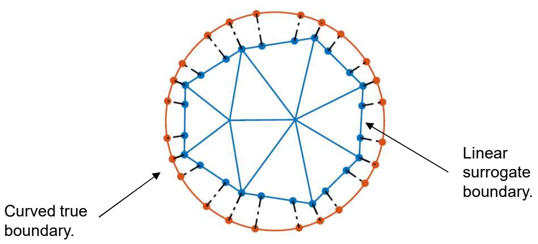
The shifted boundary method (SBM) – Part 3/3

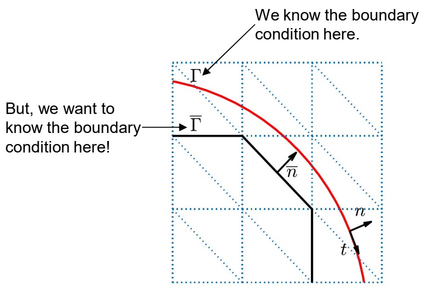
What's the trick?

Solve the problem on $\bar{\Omega}$ instead of Ω.

Problem: What to impose on $\bar{\Gamma}$? (All boundary data is given on Γ).

Solution: Taylor series expansions between functions on Γ and $\bar{\Gamma}$!





My work with the SBM

Visbech et al. (2023):

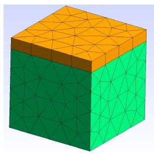
- Combing SBM with SEM (high-order expansion).
- Numerical analysis of convergence and conditioning of elliptic problems (Poisson-type).
- Dirichlet, Neumann, and Robin boundary conditions.
- The influence of different mappings and element selection.

The spectral element method (SEM):

- A high-order version of the FEM

or

- a multi-domain version of a polynomial (spectral) method.



Pros of the SEM:

- Geometrical flexibility.
- High accuracy (*p*- and *h*-convergence).
- Efficient.

See Engsig-Karup et al. (2016).

To summarize:

The **shifted boundary method** is an **unfitted/embedded/immersed approach** that:

- Naturally **represents curved/complex geometrical features** on affine/linear elements.
- Requires **no re-meshing**.
- **No *small-cut-cell-problem***.
- Can be a **high-order expansion,** e.g., through the spectral element method**.**

The incompressible NSE and the pressure problem

Continuity equation: $\nabla \cdot u = 0, \quad \text{in} \quad \Omega,$

Momentum equation: $\rho \partial_t u = -\nabla p + \mu \nabla^2 u - \rho (u \cdot \nabla) u + F, \quad \text{in} \quad \Omega \times \mathcal{T},$

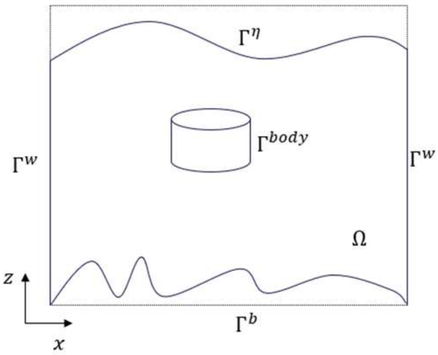
Poisson-type problem for the pressure:

$$\nabla^2 p_D = -\nabla^2 p_S + \mu \nabla \cdot \nabla^2 u - \rho \nabla \cdot (u \cdot \nabla) u, \quad \text{in} \quad \Omega,$$
$$n \cdot \nabla p_D = -n \cdot \nabla p_S + \mu n \cdot \nabla^2 u - \rho n \cdot (u \cdot \nabla) u + n \cdot F, \quad \text{on} \quad \Gamma^b \cup \Gamma^w \cup \Gamma^{\text{body}},$$
$$p_D = 0, \quad \text{on} \quad \Gamma^\eta.$$

f q

where

$$p = p_S + p_D$$
$$p_S = \rho g (\eta - z)$$



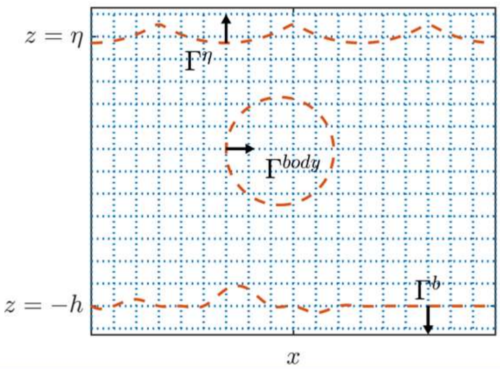
Shifted boundary polynomial correction

All details omitted: (Transformed Poisson problem, weak Aubin-type penalty formulation, shifted boundary polynomial corrections, and more). See Visbech et al. (2023) for more. **Final formulation:**

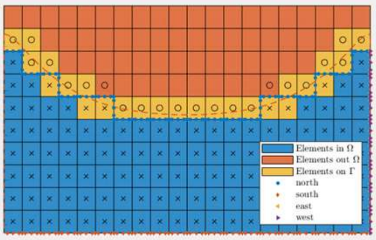
$$(\nabla p_D, \nabla v)_{\overline{\Omega}_h} + \tau (p_D(\boldsymbol{x}), v)_{\overline{\Gamma}_h^\eta} - (\nabla p_D \cdot \overline{\boldsymbol{n}}, v)_{\overline{\Gamma}_h^\eta} + \overline{n}_n (\nabla (p_D(\boldsymbol{x}) - p_D) \cdot \boldsymbol{n}, v)_{\overline{\Gamma}_h^i} - \overline{n}_t (\nabla p_D \cdot \boldsymbol{t}, v)_{\overline{\Gamma}_h^i} = (f, v)_{\overline{\Omega}_h} + \overline{n}_n (q_i, v)_{\overline{\Gamma}_h^i}$$

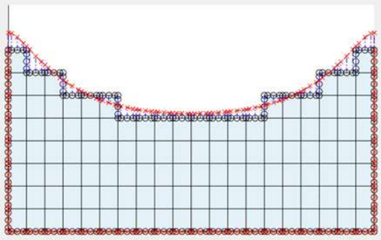
Important: This formulation (given the velocity field)

- models the pressure with unfitted boundaries (free surface, moving bodies, bathymetry, etc.)
- on a very simple Cartesian grid.

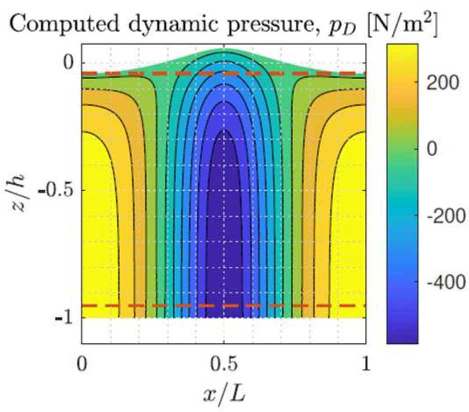


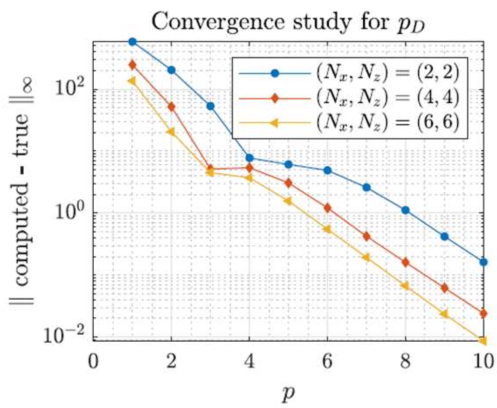
Meshes





Numerical results





Conclusion: What have we achieved?

An implementation combining:
The shifted boundary method.
The spectral element method.
A Poisson-type pressure problem within the incompressible Navier-Stokes equations.

That is to be able to simulate:
Unfitted boundaries.
With high-order accuracy.

Perspectives: What is next?

Extend the unfitted approach to the velocity field.

More elaborate cases for pure wave propagation.

Submerged and floating bodies.

Three spatial dimensions.

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