Oral presentation | Higher order methods

Higher order methods-II

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[4-C-04] Very high-order ENO schemes with multi-resolution

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Very high-order ENO schemes with multi-resolution

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1 Introduction

When we solve hyperbolic conservation laws, the difficulty lies in twofold: discontinuities may emergy in finite time even when the initial condition is sufficiently smooth, plus discontinuous solutions are usually accompanied by sophisticated structures with multi-scales. Therefore, high-order numerical schemes with excellent shock-capturing and multi-resolution properties are preferred for solving such problems. The essentially non-oscillatory (ENO) schemes [1] and weighted ENO (WENO) schemes [2, 3] are cutting-edge high-order shock-capturing schemes and achieve great success in practice.

In this study, we construct an efficient class of very high-order (up to 17th-order) essentially non-oscillatory schemes with multi-resolution (ENO-MR) for solving hyperbolic conservation laws. The candidate stencils for constructing ENO-MR schemes range from first-order one-point stencil increasingly up to the designed very high-order stencil. The proposed ENO-MR schemes adopt a very simple and efficient strategy that only requires the computation of the highest-order derivatives of a part of candidate stencils. Besides simplicity and high efficiency, ENO-MR schemes are completely parameter-free and essentially scale-invariant. Theoretical analysis and numerical computations show that ENO-MR schemes achieve designed high-order convergence in smooth regions which may contain high-order critical points (local extrema) and retain ENO property for strong shocks. In addition, ENO-MR schemes could capture complex flow structures very well.

2 Finite difference ENO-MR schemes

2.1 A semi-discretized conservative finite difference scheme

We consider the one-dimensional scalar conservation law,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, t \in [0, \infty), \tag{1}$$

in the spatial domain $[x_L, x_R]$ that is discretized into uniform intervals by $x_j = x_L + (j-1)h$ ($j = 1$ to $N+1$), where $h = (x_R - x_L)/N$. Then we can construct a semi-discretized conservative finite difference scheme as

$$\frac{du_j(t)}{dt} = \mathcal{L}(u_j(t)) = -\frac{\hat{f}_{j+1/2} - \hat{f}_{j-1/2}}{h}, \tag{2}$$

where the numerical flux $\hat{f}_{j\pm1/2}$ is an approximation of the function $\mathcal{H}(x)$ at $x_{j\pm1/2}$, which is implicitly defined as $f(u(x)) = \frac{1}{h}\int_{x-h/2}^{x+h/2} \mathcal{H}(\xi)d\xi$ [4]. This approach can be straightforwardly extended to multi-dimensional cases in a dimension-by-dimension manner.

To take account of the upwind mechanism which can improve the robustness of the scheme, we split the flux into two parts as

$$f^{\pm}(u) = \frac{1}{2}(f(u) \pm \alpha u), \tag{3}$$

where $\alpha = \max\left|\frac{df(u)}{du}\right|$ and the maximum is taken over all mesh points on one axis-aligned line.

2.2 Very high-order ENO reconstructions with multi-resolution

Define a stencil S_{j-m}^{j+n} as a set of successive intervals including I_j, i.e.,$S_{j-m}^{j+n} := \{I_{j-m}, ..., I_j, ..., I_{j+n}\}$ ($m \geq 0, n \geq 0$). For a $(2r-1)$-point scheme, there are r^2 stencils in total that can be used to reconstruct $f_{j+1/2}$. However, we only choose linearly stable stencils as candidates to guarantee stability.

The ENO-MR reconstruction procedure is as follows:

Step 1. We define a baseline smoothness indicator as

$$IS_0 = \mathrm{MIN}\left(IS_L, IS_R\right), \tag{4a}$$

with

$$IS_L = \mathrm{MAX}\left(\left|f_j - f_{j-1}\right|, \left|f_j - 2f_{j-1} + f_{j-2}\right|\right), \tag{4b}$$

$$IS_R = \mathrm{MAX}\left(\left|f_j - f_{j+1}\right|, \left|f_j - 2f_{j+1} + f_{j+2}\right|\right). \tag{4c}$$

Step 2. We define smoothness indicators for S_{j-m}^{j+n} as

$$IS_{j-m}^{j+n} = \left|\frac{d^{m+n} P_{j-m}^{j+n}(x)}{dx^{m+n}}\right| h^{m+n}, \tag{5}$$

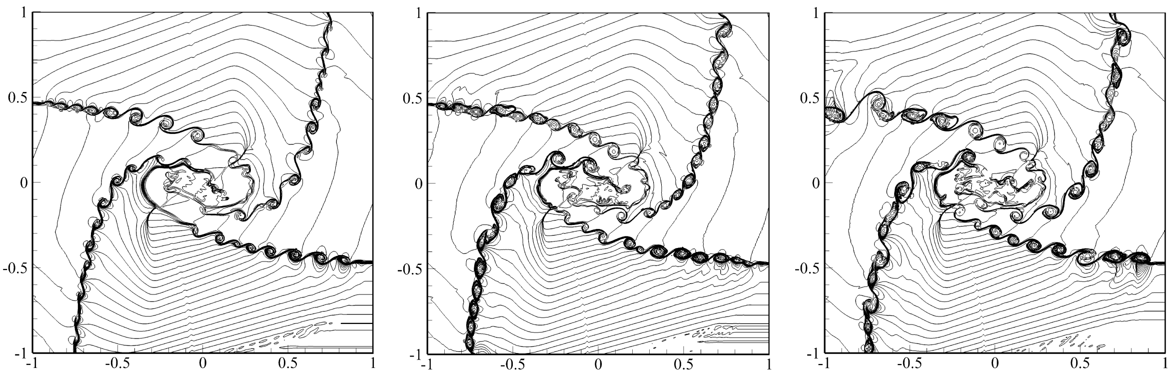
where $P_{j-m}^{j+n}(x)$ is the polynomial reconstructed on S_{j-m}^{j+n}.

Step 3. We compare the smoothness indicators of candidate stencils in sequence from high-order to low-order with the baseline smoothness indicator.

Step 3.1. If any IS_{j-m}^{j+n} ($m \geq 1$ and $n \geq 1$) is smaller than the baseline IS_0, we directly use the reconstructed flux on S_{j-m}^{j+n}.

Step 3.2. If all IS_{j-m}^{j+n} ($m \geq 1$ and $n \geq 1$) are larger than the baseline IS_0, we use the minmod function to select a low-order stencil from $\{S_j^{j+1}, S_{j-1}^{j}, S_j^{j}\}$.

Fig. 1 shows density contours of the 2D Riemann problem at $t = 1$ calculated by 5th-, 9th-, and 17th-order ENO-MR schemes with 801×801 mesh points.



(a) ENO-MR5 (b) ENO-MR9 (c) ENO-MR17

Figure 1: Density contours of the 2D Riemann problem at $t = 1$ calculated by ENO-MR schemes with 801×801 mesh points.

References

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