Viscoplastic free-surface flows: the Herschel-Bulkley case.

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Abstract: In this talk, we will describe consistant numerical methods for power-law viscoplastic free-surface flows. After describing a shallow-water asymptotics of a 3D Navier-Stokes-Herschel-Bulkley model with free surface, we will end up with a model which has various mathematical difficulties. We will here discuss validity of various approaches to handle optimization problems arising from the variational inequalities associated to the model, as well as their coupling with finite-volume discretization. Several numerical tests will be shown, including a comparison with an analytic solution, to confirm the well balanced property and the ability to cope with the various rheological regimes associated with the Herschel-Bulkley constitutive law.

Keywords: variational inequality, finite volume, well balanced, Herschel-Bulkley, viscous shallow water, avalanche.

Problem Statement and Results

We are here interested in the derivation of an integrated *Herschel-Bulkley* model for shallow flows, as well as in the design of a numerical algorithm to solve the resulting equations. The goal being to simulate the evolution of thin sheet of *viscoplastic materials* on inclined planes and, in particular, being able to compute the evolution from dynamic to stationary states. Among numerous models used to describe the rheology of viscoplastic materials, Bingham (linear model with plasticity, [1]) and Herschel-Bulkley (power law model with plasticity, [2]) models are probably the most iconic. The Herschel-Bulkley model is expressed as :

$$\begin{aligned} \sigma' &= \left(2^{\wp} K |D(u)|^{\wp-1} + \tau_c \frac{1}{|D(u)|}\right) D(u), && \text{if } |D(u)| \neq 0, \\ |\sigma'| &\leq \tau_c, && \text{if } |D(u)| = 0, \end{aligned} \tag{1}$$

with K the constant consistency, τ_c the yield stress and $|D(u)|$ is the second invariant of the rate of strain. This model can be seen as a generalization of the Bingham model which is retrieved from (1) by taking $\wp = 1$. On the one hand, Bingham model is the simplest model when it comes to describe plasticity. On the other hand, if we take $\tau_c = 0$ and $\wp < 1$, we end up with the classical power-law (shear-thinning) model. Evidently, if $\tau_c = 0$ and $\wp = 1$, (1) leads to the classic Navier-Stokes equations. It appears in recent years that to gain insight into the dynamic behavior of finite volumes of viscoplastic materials down inclined planes, Herschel-Bulkley model has attracted growing attention both from the experimental and theoretical viewpoints, see for instance [3] and references therein. Starting from a 3D incompressible fluid modelled by the

Navier-Stokes equations, together with the Herschel-Bulkley constitutive law (1), and with a free surface, we introduce its formulation as a variational inequality and derive a shallow water asymptotic of this system. For sake of brevity (cf. [4] for details), we here give a 1D version (which nonetheless contains aforementioned difficulties) :

$$\frac{\partial H}{\partial t} + \frac{\partial (H\boldsymbol{V})}{\partial x} = 0, \tag{2}$$

$$\begin{aligned}
&\int_0^L H\Big(\partial_t \boldsymbol{V}(\boldsymbol{\Psi} - \boldsymbol{V}) + \frac{1}{2}\partial_x(\boldsymbol{V}^2)(\boldsymbol{\Psi} - \boldsymbol{V})\Big)dx + \int_0^L \tau_c\sqrt{2}H\Big(|\partial_x(\boldsymbol{\Psi})| - |\partial_x(\boldsymbol{V})|\Big)dx \\
&+ \int_0^L \tilde{\alpha}\boldsymbol{V}(\boldsymbol{\Psi} - \boldsymbol{V})dx + \int_0^L 2^{\frac{3+\wp}{2}} H\frac{\nu_1}{|\partial_x \boldsymbol{V}|^{1-\wp}}\partial_x(\boldsymbol{V})\partial_x(\boldsymbol{\Psi} - \boldsymbol{V})dx \\
&\geq -g\sin\theta\int_0^L H(\boldsymbol{\Psi} - \boldsymbol{V}) + g\frac{\cos\theta}{2}\int_0^L (H)^2(\partial_x\boldsymbol{\Psi} - \partial_x\boldsymbol{V})dx, \quad \forall\boldsymbol{\Psi}.
\end{aligned} \tag{3}$$

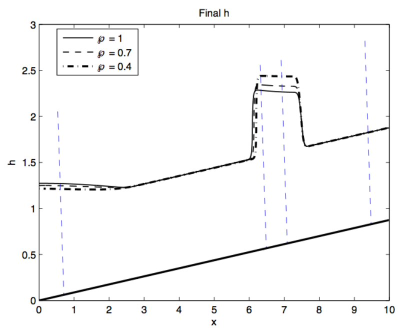

Figure 1: Stationary states of an academic avalanche for various Herschel-Bulkley fluids.

We design a coupled duality methods / finite volume scheme which fully takes into account *both* the *threshold* of plasticity and the *power law*. The overall method is well balanced. All these characteristics lead to a scheme which is able to compute the evolution to stationary solutions which can arise in these type of flow (thanks to plastic behaviour). It is quite a remarkable feature of the present approach since many of the numerical methods presented in the literature use a so called regularization of the constitutive law, skipping the mathematical difficulty induced by plasticity and making them unable to compute stationary states (the material can not become rigid). The overall method is able to catch the various rheological behaviours of Herschel-Bulkley and allows us to tackle practical situations, such as the one presented in [3].

References

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