Very-High-Order Conservative Discretization of Diffusive Terms with Variable Viscosity

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Abstract: We study very-high-order conservative discretizations for diffusive terms with variable viscosity, which are present in the compressible Navier-Stokes equations, using viscous fluxes at cell-interfaces. We show that the proposed approach yields $O(\Delta x^2)$ accuracy on the stencil $\{i-s, \cdots, i, \cdots, i+s\}$, thus improving upon previous proposals which are $O(\Delta x^{3/2})$ on the same stencil. The extension of the method to 2-D and 3-D regular cartesian grids is described. Several typical 1-D and 2-D computational examples substantiate the accuracy of the method for test problems and for DNS of turbulent flows using the 3-D compressible Navier-Stokes equations.

Keywords: High-Order Schemes, Viscous Terms, Diffusion Equation, Compressible Navier-Stokes.

1 Introduction

Very-high-order accuracy is essential in several practical applications such as DNS of compressible turbulence [3]. Whereas several very-high-order approaches for the discretization of convective terms have been developed [2], the conservative discretization of the diffusive (viscous) terms has received less attention. The popular compact scheme of Lele [4] is nonconservative. Zingg et al. [6] developed an alternative $O(\Delta x^4)$-accurate approximation of $(\mu(x) \frac{\partial u}{\partial x}(x))^\cdot$, and Shen et al. [5] developed an alternative $O(\Delta x^4)$-accurate conservative formulation on the same stencil. It is straightforward to generalize these approaches to higher-order using larger stencils, obtaining an $O(\Delta x^2\cdot\frac{\partial}{\partial x})$ method on the stencil $\{i-s, \cdots, i+s\}$. In the present work we develop an $O(\Delta x^2\cdot\frac{\partial}{\partial x})$ method on the stencil $\{i-s, \cdots, i+s\}$ twice more accurate.

2 Present Approach

To discretize $(\mu(x) \frac{\partial u}{\partial x}(x))^\cdot$ on a homogeneous grid $x := x - 1 + (i-1)\Delta x$ we define the numerical flux $\hat{F}_{(\mu u',i,s-1,s)}(\cdot \cdot \cdot)$ on the stencil $\{i-s+1, \cdots, i+s\}$ satisfying

$$\frac{1}{\Delta x} (\hat{F}_{(\mu u',i,s-1,s)}(\cdot \cdot \cdot) - \hat{F}_{(\mu u',i,s,s)}(\cdot \cdot \cdot)) = O(\Delta x^2)$$ (1a)

Let $p_{i,M-\cdot,M_+}(x_i, \Delta x; f)$ be the Lagrange interpolating polynomial of $f : \mathbb{R} \to \mathbb{R}$ on the stencil $\{i-M, \cdots, i+M\}$ and $p_{R_{i,M-\cdot,M_+}}(x_i, \Delta x; f)$ the corresponding reconstructing polynomial (1) which approximates the function $h : \mathbb{R} \to \mathbb{R}$, whose cell-averages are equal to $f(x)$ $(f(x) = \int_{x}^{x+\frac{\Delta x}{2}} h(x+\zeta \Delta x) d\zeta \forall x)$. Then we can show analytically and verify computationally (Fig. 1) that the required numerical flux is

$$\hat{F}_{(\mu u',i,s-1,s)}(\cdot \cdot \cdot) := p_{R_{i,s-1,s}}(x_i+\frac{\Delta x}{2}; x_i; \Delta x; [p_{i,s-1,s}(x_i; x_i; \Delta x; \mu)p'_{i,s-1,s}(x_i; x_i; \Delta x; f)])$$ (1b)
Figure 1: (Left) Error of the present approximation for the computation of \((\mu u')'\) as a function of grid refinement and comparison with previous approaches [5, 6]. (Center) High-Mach-number Couette flow testcase. (Right) Application to DNS of compressible channel flow.

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\hat{F}_{(\mu u',1,s-1,s),1+} = \frac{1}{\Delta x_p} \sum_{p=-s+1}^{s} \mu_{i+p} \sum_{q=-s+1}^{s} \left( \sum_{\ell=-s+1}^{s} \alpha_{R_1,s-1,s,\ell} \alpha_{I_2,s-1,s,q}(\ell) \right) \alpha_{i+q} \quad (1c)
\]

The improvement upon previous approaches [5, 6] comes from the fact that we do not reconstruct fluxes from interpolatory approximations of the product \(\mu u'\) at half-points, but instead at the integer points of the stencil. At boundary-points, we use biased stencils recovering global \(O(\Delta x^{2s-1})\) accuracy. The method is extended to 2-D and 3-D using the usual linewise approach [3, 5, 6]. Typical applications presented in the complete paper include:
1) Nonisothermal flow of glycerol (whose viscosity varies exponentially with temperature \(T\))
2) Compressible laminar Couette flow (Fig. 1)
3) 2-D diffusion equation
4) DNS computations (Fig. 1).

3 Conclusion and Future Work

The present work defines numerical fluxes for very-high-order conservative discretization of \((\mu u')'\), applicable to the viscous terms of the Navier-Stokes equations. Future work includes a least-squares genuinely multidimensional approach applicable to arbitrary unstructured grids and the development of WENO discretizations of these terms for flows with discontinuities.

References